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DISCUSSION ON THE PAPER: IL'IUSHIN'S POSTULATE AND RESULTING THERMODYNAMIC CONDITIONS ON ELASTO-PLASTIC COUPLING[1]

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Il'iushin's postulate†

$$W = \int_p \sigma_{ij}\dot{\epsilon}_{ij}\, \mathrm{d}t \geq 0 \tag{1}$$

was recently exploited in a paper[1] by Y. F. Defalias to derive restrictions on the elasto-plastic coupling in the thermodynamic theory of rate-independent elasto-plastic materials. The paper contains a clear introduction giving a sharp insight into the problem there considered. This, of course, facilitates the criticism of the new results presented in [1] and I must, therefore, apologize should my comments which follow be not expressive of the help which that introduction gave me.

The general thermodynamic theory is formulated in [1] in terms of an undefined set of internal plastic variables q_N; the main results, however, are derived by particularizing this set in the form

$$\{q_N\} = \{E^p_{KL}, q_n\}. \tag{2}$$

Here E^p_{KL} is the plastic strain tensor defined by

$$E_{KL} = E^e_{KL} + E^p_{KL}, \tag{3}$$

where E_{KL} and E^e_{KL} are two Lagrangian strain tensors representing total deformation and elastic deformation, respectively. By introducing a free energy function of the form

$$\psi = \hat{\psi}(E_{KL}, \theta, E^p_{KL}, q_n), \tag{4}$$

and a yield surface of the kind

$$F(E_{KL}, \theta, E^p_{KL}, q_n) = 0 \tag{5}$$

(*strain space description*), Defalias deduced from inequality (1) and from the second principle of thermodynamics the central relation of his paper

$$\frac{\partial}{\partial q_N}(\hat{\psi}_M - \hat{\psi}_T) r_N \geq 0. \tag{6}$$

To obtain this result, he considered a closed cycle of deformation in which the material is brought from a point T on the yield surface to a point P in the strain space outside of the yield surface and at infinitesimal distance from it. If the trivial case of yield surface of infinitesimal dimensions is excluded, the deformation increment which brings the material from a point M within the yield surface to the point T is, in general, finite with respect to the plastic deformation

†In this discussion the notation of [1] will be consistently followed. The reader is referred to [1] for a definition of the quantities introduced here.

increment from T to P. This implies that to derive relation (6) a theory valid in the range of finite deformations must be considered. Indeed, though not explicitly made, the allowance of finite deformations appears to be obvious in paper [1], which, otherwise, would lose much of its relevance. However, it is well known (see, e.g. [2, p. 22] and the references there quoted) that when a plastic strain measure such as that defined by (3) is adopted, then the plastic strain tensor E^p_{KL} is influenced by the elastic strain tensor E^e_{KL}, even if no plastic flow occurs (as, for instance, during unloading processes). It turns out that in the strain space description of plasticity adopted in [1] it is not true that $E^p_{KL} = 0$ for unloading processes or for neutral processes and, therefore the condition

$$\dot{E}^p_{KL} = 0 \quad \text{for} \quad L = \frac{\partial F}{\partial E_{KL}} \dot{E}_{KL} + \frac{\partial F}{\partial \theta} \dot{\theta} \leq 0 \tag{7}$$

introduced in [1] is not valid.† Since, moreover, it is known that the yield surface is unaffected by deformation processes which occur within it, and since during such processes the tensor E^p_{KL} can vary, it follows that the explicit dependence on E^p_{KL} must be dropped from (5). It will be shown in a moment that the second principle of thermodynamics requires that the explicit dependence on E^p_{KL} must be dropped also from (4) when ψ is relevant to points inside of the yield surface. This implies that in relation (6) the derivative of $\hat{\psi}_M$ with respect to E^p_{KL} must vanish, which, in turn, makes much of the results contained in Section 5 of [1] insignificant.

The second principle of thermodynamics postulated in [1] can be expressed by

$$-\dot{\psi} + S_{KL}\dot{E}_{KL} \geq 0 \tag{8}$$

when isothermal processes are considered. Any closed isothermal deformation cycle which occurs on the yield surface is a closed thermodynamic cycle because at the end of the cycle the material recovers the same state it possesses at the beginning of the cycle; no plastic flow occurring in the material during such a cycle. For this cycle the relation

$$\oint S_{KL}\dot{E}_{KL}\, \mathrm{d}t = 0 \tag{9}$$

holds true. Indeed, by integrating (8) along the cycle we get

$$\oint S_{KL}\dot{E}_{KL}\, \mathrm{d}t \geq 0, \tag{10}$$

because ψ assumes the same values at the beginning and at the end of the cycle and, hence, $\oint\dot{\psi}\, \mathrm{d}t = 0$. The inequality sign in (10) must be excluded since, otherwise, by reverting the cycle we could arrive at a contradiction with (8). Hence equation (9) follows. From (9) and from relation $(8)_1$ of [1] it follows that

$$\oint \frac{\partial\hat{\psi}}{\partial E_{KL}} \dot{E}_{KL}\, \mathrm{d}t = 0. \tag{11}$$

Since $\oint\dot{\psi}\, \mathrm{d}t = 0$, it can be deduced from (11) and (4) that

$$\oint \dot{\psi}\, \mathrm{d}t = \oint \left(\frac{\partial\hat{\psi}}{\partial E^p_{KL}} \dot{E}^p_{KL} + \frac{\partial\hat{\psi}}{\partial q_n} \dot{q}_n \right) \mathrm{d}t = 0. \tag{12}$$

In this relation, however, $\dot{q}_n \equiv 0$ because no plastic flow occurs in this cycle. It follows, therefore, that

$$\oint \frac{\partial\hat{\psi}}{\partial E^p_{KL}} \dot{E}^p_{KL}\, \mathrm{d}t = \int_A^{A'} \frac{\partial\hat{\psi}}{\partial E^p_{KL}} \dot{E}^p_{KL}\, \mathrm{d}t + \int_{A'}^{A} \frac{\partial\hat{\psi}}{\partial E^p_{KL}} \dot{E}^p_{KL}\, \mathrm{d}t = 0 \tag{13}$$

†For the same reason the analysis by P. M. Naghdi and J. A. Trapp in Section 3 of Ref. 17 of [1] does not appear to be correct, in general.

for every pair of points A and A' of the considered cycle. From (13) and from the fact that $-(\partial\hat{\psi}/E^p_{KL})\dot{E}^p_{KL} \geq 0$ (see eqn (9) of [1]) it follows that

$$\frac{\partial\hat{\psi}}{\partial E^p_{KL}}\dot{E}^p_{KL} = 0 \tag{14}$$

for every isothermal deformation process which occurs on the yield surface. By repeating the above analysis for a deformation cycle occurring inside of the yield surface, we get that

$$\frac{\partial\hat{\psi}}{\partial E^p_{KL}}\dot{E}^p_{KL} = 0, \tag{15}$$

which clearly implies that $\partial\hat{\psi}_M/\partial E^p_{KL} = 0$ for every point M within the yield surface. As a consequence of this, relation (25) of [1] reduces to

$$\frac{\partial}{\partial q_n}\hat{\psi}_M r_n - \frac{\partial}{\partial q_N}\hat{\psi}_T r_N \geq 0 \tag{16}$$

(note the difference in capital and small n). In view of (15) and in view of the fact that eqn (26) of [1] is not valid in the range of finite deformations, much of the analysis in Section 5 of [1] fails to be correct. Observe, moreover, that eqn (12) of [1] seems likely to be wrong, because, as it stands, it would imply changes in the plastic internal variables for processes in the elastic range (since, in general, $\dot{E}^p_{KL} \neq 0$ for these processes). However, no derivation of this equation is given in [1].

Finally, it may be useful to remark that eqn (14), which has been derived here, may be used to relate the yield surface to the free energy function. Since (14) is valid for isothermal processes belonging to the yield surface, the tensor $\dot{E}^p_{KL}$ which appears in it is tangent to this surface. Equation (14) states, therefore, that the tensor $\partial\hat{\psi}/\partial E^p_{KL}$ is normal to the yield surface, when $\theta = \text{const}$. Since the tensor $\partial F/\partial E_{KL}$ is normal to the yield surface for $\theta = \text{const}$, it follows that

$$\frac{\partial\hat{\psi}}{\partial E^p_{KL}} = \lambda\frac{\partial F}{\partial E_{KL}}, \tag{17}$$

where λ is a scalar. When E^p_{KL} is the only internal plastic variable, then from (9) of [1] we get

$$-\frac{\partial\hat{\psi}}{\partial E^p_{KL}}\dot{E}^p_{KL} \geq 0 \tag{18}$$

for increments $\dot{E}^p_{KL}\,dt$ of plastic deformation which lead the material outside of the yield surface. Since $\partial F/\partial E_{KL}$ points out of the yield surface, for the above increments of plastic deformation the relation

$$\frac{\partial F}{\partial E_{KL}}\dot{E}^p_{KL} \geq 0 \tag{19}$$

holds. From (19), (17) and (18) it follows, therefore, that at the yield surface the equation

$$\frac{\partial\psi}{\partial E^p_{KL}} = -k^2\frac{\partial F}{\partial E_{KL}} \tag{20}$$

is valid, k being a scalar.

REFERENCES

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